6. Let

$$\mathbb{A} = \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix}.$$

Find the eigenvalues of $\mathbb{A}$ and prove that $\mathbb{A}$ is diagonalizable, *i.e.*, there is an invertible matrix $\mathbb{P}$ such that $\mathbb{P}^{-1}\mathbb{A}\mathbb{P}$ is diagonalizable. Furthermore, $\mathbb{P}$ can be chosen to be an orthogonal matrix, *i.e.*, a matrix whose columns are unit vectors and whose distinct columns are orthogonal to each other.

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} 1-t & -\frac{3}{2} \\ -\frac{3}{2} & 1-t \end{vmatrix} = (1-t)^2 - \frac{9}{4}$$

Roots are: $-\frac{1}{2}$ and $\frac{5}{2}$.

Notice also: $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-1/2\\-1/2\end{bmatrix} = -1/2\begin{bmatrix}1\\1\end{bmatrix}$, $A\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}5/2\\-5/2\end{bmatrix} = \frac{5}{2}\begin{bmatrix}1\\-1\end{bmatrix}$.

$P = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$ is an orthogonal matrix, $P^{-1}AP = \begin{bmatrix}-1/2 & 0\\0 & 5/2\end{bmatrix}$

Recall: Thm If A is a symmetric $n \times n$ matrix, then A is diagonalizable by an orthogonal matrix P, i.e. $P^{-1}AP$ is diagonal. Equivalently, $\mathbb{R}^n$ has an [redacted] orthogonal basis of eigenvectors for A.

7. Let A be a 3×3 symmetric matrix with real entries, and have eigenvalues $\lambda_1 = 0$, $\lambda_2 = 6$, and $\lambda_3 = 3$. Let $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$ be corresponding eigenvectors (normalized to have length one).

(a) Since A is symmetric and λ_1, λ_2, and λ_3 are all different, it follows that

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = \underline{0}$$

(b) The 3×3 matrix $P = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3]$ satisfies

$$P^{\mathrm{T}}P = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$ (fill in the entries of this 3×3 matrix).

(c) Let P be the matrix of normalized eigenvectors from **(b)**. Then $A = PDP^{\mathrm{T}}$, where

$$D = \begin{bmatrix}0 & 0 & 0\\0 & 6 & 0\\0 & 0 & 3\end{bmatrix}$$ (fill in the entries of this 3×3 matrix).

(d) The characteristic polynomial of A is $\underline{(-t)\cdot(6-t)\cdot(3-t)}$.