

DAY 2

Problem Form

IMO 2003 JAPAN



Country Number: _____

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Version: English

Second day
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Problem 4. Let $ABCD$ be a cyclic quadrilateral. Let P , Q and R be the feet of the perpendiculars from D to the lines BC , CA and AB respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ meet on AC .

Problem 5. Let n be a positive integer and x_1, x_2, \dots, x_n be real numbers with $x_1 \leq x_2 \leq \dots \leq x_n$.

(a) Prove that

$$\left(\sum_{i=1}^n \sum_{j=1}^n |x_i - x_j| \right)^2 \leq \frac{2(n^2 - 1)}{3} \sum_{i=1}^n \sum_{j=1}^n (x_i - x_j)^2.$$

(b) Show that equality holds if and only if x_1, x_2, \dots, x_n is an arithmetic sequence.

Problem 6. Let p be a prime number. Prove that there exists a prime number q such that for every integer n , the number $n^p - p$ is not divisible by q .

Time allowed: four and a half hours.
Each problem is worth seven points.