21. **Answer (A):** The product of the zeros of $f$ is $c/a$, and the sum of the zeros is $-b/a$. Because these two numbers are equal, $c = -b$, and the sum of the coefficients is $a + b + c = a$, which is the coefficient of $x^2$. To see that none of the other choices is correct, let $f(x) = -2x^2 - 4x + 4$. The zeros of $f$ are $-1 \pm \sqrt{3}$, so the sum of the zeros, the product of the zeros, and the sum of the coefficients are all $-2$. However, the coefficient of $x$ is $-4$, the $y$-intercept is 4, the $x$-intercepts are $-1 \pm \sqrt{3}$, and the mean of the $x$-intercepts is $-1$.

22. **Answer (D):** If $n \le 2007$, then $S(n) \le S(1999) = 28$. If $n \le 28$, then $S(n) \le S(28) = 10$. Therefore if $n$ satisfies the required condition it must also satisfy

$$n \ge 2007 - 28 - 10 = 1969.$$

In addition, $n, S(n)$, and $S(S(n))$ all leave the same remainder when divided by 9. Because 2007 is a multiple of 9, it follows that $n, S(n)$, and $S(S(n))$ must all be multiples of 3. The required condition is satisfied by 4 multiples of 3 between 1969 and 2007, namely 1977, 1980, 1983, and 2001.

Note: There appear to be many cases to check, that is, all the multiples of 3 between 1969 and 2007. However, for $1987 \le n \le 1999$, we have $n + S(n) \ge 1990 + 19 = 2009$, so these numbers are eliminated. Thus we need only check 1971, 1974, 1977, 1980, 1983, 1986, 2001, and 2004.

23. **Answer (A):** Let $A = (p, \log_a p)$ and $B = (q, 2\log_a q)$. Then $AB = 6 = |p - q|$. Because $\overline{AB}$ is horizontal, $\log_a p = 2\log_a q = \log_a q^2$, so $p = q^2$. Thus $|q^2 - q| = 6$, and the only positive solution is $q = 3$. Note that $C = (q, 3\log_a q)$, so $BC = 6 = \log_a q$, from which $a^6 = q = 3$ and $a = \sqrt[6]{3}$.

24. **Answer (D):** Note that $F(n)$ is the number of points at which the graphs of $y = \sin x$ and $y = \sin nx$ intersect on $[0, \pi]$. For each $n$, $\sin nx \ge 0$ on each interval $[(2k-2)\pi/n, (2k-1)\pi/n]$ where $k$ is a positive integer and $2k - 1 \le n$. The number of such intervals is $n/2$ if $n$ is even and $(n+1)/2$ if $n$ is odd. The graphs intersect twice on each interval unless $\sin x = 1 = \sin nx$ at some point in the interval, in which case the graphs intersect once. This last equation is satisfied if and only if $n \equiv 1 \pmod 4$ and the interval contains $\pi/2$. If $n$ is even, this count does not include the point of intersection at $(\pi, 0)$. Therefore $F(n) = 2(n/2) + 1 = n + 1$ if $n$ is even, $F(n) = 2(n+1)/2 = n + 1$ if $n \equiv 3 \pmod 4$, and $F(n) = n$ if $n \equiv 1 \pmod 4$. Hence

$$\sum_{n=2}^{2007} F(n) = \left(\sum_{n=2}^{2007} (n+1)\right) - \left\lfloor \frac{2007-1}{4} \right\rfloor = \frac{(2006)(3+2008)}{2} - 501 = 2{,}016{,}532.$$



25. **Answer (E):** For each positive integer $n$, let $S_n = \{k : 1 \le k \le n\}$, and let $c_n$ be the number of spacy subsets of $S_n$. Then $c_1 = 2$, $c_2 = 3$, and $c_3 = 4$. For $n \ge 4$, the spacy subsets of $S_n$ can be partitioned into two types: those that contain $n$ and those that do not. Those that do not contain $n$ are precisely the spacy subsets of $S_{n-1}$. Those that contain $n$ do not contain either $n-1$ or $n-2$ and are therefore in one-to-one correspondence with the spacy subsets of $S_{n-3}$. It follows that $c_n = c_{n-3} + c_{n-1}$. Thus the first twelve terms in the sequence $(c_n)$ are 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, and there are $c_{12} = 129$ spacy subsets of $S_{12}$.

OR

Note that each spacy subset of $S_{12}$ contains at most 4 elements. For each such subset $a_1, a_2, \ldots, a_k$, let $b_1 = a_1 - 1$, $b_j = a_j - a_{j-1} - 3$ for $2 \le j \le k$, and $b_{k+1} = 12 - a_k$. Then $b_j \ge 0$ for $1 \le j \le k+1$, and

$$b_1 + b_2 + \cdots + b_{k+1} = 12 - 1 - 3(k-1) = 14 - 3k.$$

The number of solutions for $(b_1, b_2, \ldots, b_{k+1})$ is $\binom{14-2k}{k}$ for $0 \le k \le 4$, so the number of spacy subsets of $S_{12}$ is

$$\binom{14}{0} + \binom{12}{1} + \binom{10}{2} + \binom{8}{3} + \binom{6}{4} = 1 + 12 + 45 + 56 + 15 = 129.$$