

## Mathematical Proofs of the Binomial Theorem

### Question

Use mathematical induction to prove the binomial theorem. (The binomial theorem is  $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$ .)

### Answer

Base Case:

$$(x + y)^0 = \sum_{k=0}^0 \binom{0}{k} x^{0-k} y^k = \binom{0}{0} x^0 y^0 = 1 \cdot 1 = 1$$

$$(x + y)^1 = \sum_{k=0}^1 \binom{1}{k} x^{1-k} y^k = \binom{1}{0} x^1 y^0 + \binom{1}{1} x^0 y^1 = x + y$$

Inductive Step:

$$(x + y)^{n+1} = (x + y)^n (x + y) = \left( \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k \right) (x + y)$$

$$= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k x + \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k y$$

Inductive Hypothesis:

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

$$(x + y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k$$

Inductive Step 2:

$$(x + y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k$$

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Inductive Step 3:

$$(x + y)^{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} x^{n+1-k} y^k$$

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### Additional Information