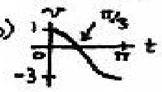
Math 125 C,D

Winter 2003

Midterm I Solutions

1. (a) $v(t)=\int a(t)\,dt = \int(-2\sin t)\,dt = 2\cos t + C$ (by guess & check)

$1 = v(0) = 2\cos(0)+C = 2+C \Rightarrow C=-1 \Rightarrow \boxed{v(t) = 2\cos t - 1}$

(b) 

$v(t) = 2(\cos t - \frac{1}{2}) \ge 0$ when $\cos t \ge \frac{1}{2} \Leftrightarrow 0\le t\le \frac{\pi}{3}$ (for $0\le t\le \pi$)

So $|v(t)| = \begin{cases} v(t) = 2\cos t - 1 & \text{for } 0\le t\le \frac{\pi}{3} \\ -v(t) = 1-2\cos t & \text{for } \frac{\pi}{3}\le t\le \pi \end{cases}$

Tot. dist. $= \int_0^{\pi} |v(t)|\,dt = \int_0^{\pi/3}(2\cos t - 1)\,dt + \int_{\pi/3}^{\pi}(1-2\cos t)\,dt$

$= (2\sin t - t)\Big|_0^{\pi/3} + (t-2\sin t)\Big|_{\pi/3}^{\pi} = (\sqrt{3}-\frac{\pi}{3}) - 0 + (\pi - 0) - (\frac{\pi}{3}-\sqrt{3}) = \boxed{2\sqrt{3}+\frac{\pi}{3}}$

2. (a) $\int x e^{x^2}\sec^2(e^{x^2})\,dx = \int (\sec^2 u)\cdot\frac{1}{2}\,du = \frac{1}{2}\tan u + C = \boxed{\frac{1}{2}\tan(e^{x^2}) + C}$

$\left[\begin{array}{l} u = e^{x^2} \\ du = 2xe^{x^2}\,dx \end{array}\right.$

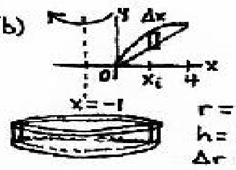
(b) $\int_1^{\sqrt{e}} \frac{\cos(\pi \ln x)}{x}\,dx = \int_0^{1/2}\cos(\pi u)\,du = \frac{1}{\pi}\sin(\pi u)\Big|_0^{1/2} = \frac{1}{\pi}(1-0) = \boxed{\frac{1}{\pi}}$

$\left[\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}\right.$ $\quad x = \sqrt{e} \Leftrightarrow u = \ln\sqrt{e} = \frac{1}{2}$, $\quad x = 1 \Leftrightarrow u = \ln 1 = 0$

3. (a) Intersections: $x = 2\sqrt{x} \Leftrightarrow x^2 = 4x \Leftrightarrow x^2 - 4x = 0 \Leftrightarrow x(x-4) = 0 \Leftrightarrow x = 0, 4$

$A = \int_a^b (f_2(x) - f_1(x))\,dx = \int_0^4 (2\sqrt{x} - x)\,dx$

$= \left(\frac{4}{3}x^{3/2} - \frac{1}{2}x^2\right)\Big|_0^4 = \left(\frac{4}{3}\cdot 8 - \frac{1}{2}\cdot 16\right) - (0) = \boxed{\frac{8}{3}}$

(b) 

Shells. Volume of shell near $x_i \approx 2\pi r h \Delta r \approx 2\pi(x_i+1)(2\sqrt{x_i} - x_i)\Delta x$

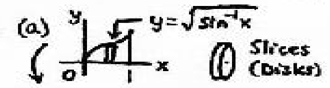
$r = x_i + 1$

$h = 2\sqrt{x_i} - x_i$

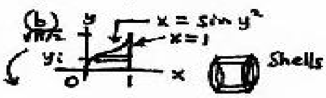
$\Delta r = \Delta x$

Volume $\approx \sum_{0\le x_i\le 4} 2\pi(x_i+1)(2\sqrt{x_i}-x_i)\,\Delta x$

$\boxed{\text{Volume} = \int_0^4 2\pi(x+1)(2\sqrt{x}-x)\,dx}$

4. (a) 

$V = \int_a^b \pi(f(x))^2\,dx = \int_0^1 \pi\left(\sqrt{\sin^{-1}x}\right)^2 dx = \boxed{\int_0^1 \pi\sin^{-1}x\,dx}$

(b) 

Solve for x: $y^2 = \sin^{-1}x$, $x = \sin(y^2)$. Upper limit: $y = \sqrt{\sin^{-1}(1)} = \sqrt{\frac{\pi}{2}}$

$V = \int_c^d 2\pi y\,(g_2(y) - g_1(y))\,dy = \int_0^{\sqrt{\pi/2}} 2\pi y\,(1-\sin(y^2))\,dy$

$\left[\begin{array}{l} u = y^2 \\ du = 2y\,dy \end{array}\right.$ $\quad y = \sqrt{\pi/2} \Leftrightarrow u = \pi/2$, $\quad y = 0 \Leftrightarrow u = 0$

$= \int_0^{\pi/2}\pi(1-\sin u)\,du$

$= \pi(u + \cos u)\Big|_0^{\pi/2} = \pi\left[\left(\frac{\pi}{2}+0\right) - (0+1)\right] = \boxed{\frac{\pi^2}{2} - \pi}$

5. (a) $s(\frac{1}{2}) = \int_0^{\pi/2}\sqrt{4\sin^2 x + 1}\,dx$. Let $a = 0$, $b = \frac{\pi}{2}$, $n = 3$, $\Delta x = \frac{b-a}{n} = \frac{\pi}{6}$; $x_0 = 0$, $x_1 = \frac{\pi}{6}$, $x_2 = \frac{\pi}{3}$, $x_3 = \frac{\pi}{2}$

$R_3 = \Delta x\,(f(x_1) + f(x_2) + f(x_3)) = \frac{\pi}{6}\left(\sqrt{4\sin^2\frac{\pi}{6}+1} + \sqrt{4\sin^2\frac{\pi}{3}+1} + \sqrt{4\sin^2\frac{\pi}{2}+1}\right)$

$= \boxed{\frac{\pi}{6}(\sqrt{2} + 2 + \sqrt{5})} \approx 2.95848$

(b) Let $G(u) = \int_0^u \sqrt{4\sin^2 x + 1}\,dx$. Then $G'(u) = \sqrt{4\sin^2 u + 1}$ (by F.T.C. Part 1)

Since $s(t) = G(\pi t)$, $v(t) = \frac{ds}{dt} = G'(\pi t)\cdot\pi = \boxed{\pi\sqrt{4\sin^2(\pi t) + 1}}$