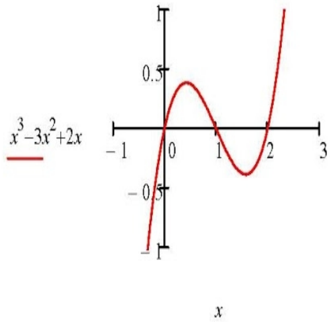
Given the function $f(x) = x^3 - 3x^2 + 2x$

(a) Find the equation of the line that is tangent to curve $y = f(x)$ at point $(a, f(a))$.

(b) Find the range of values of p for which the curve has 3 different tangents passing through point $(0, p)$



Solution :

(a)

$f'(x) = 3x^2 - 6x + 2$ at point $(a, f(a))$ the gradient of the tangent is $3(a)^2 - 6a + 2$

The equation of the tangent is

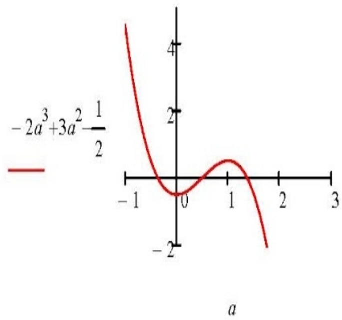
$y - f(a) = (3a^2 - 6a + 2)(x - a) \Rightarrow$ using $y - y_1 = m(x - x_1)$

$y - (a^3 - 3a^2 + 2a) = (3a^2 - 6a + 2)(x - a)$

$y - (a^3 - 3a^2 + 2a) = (3a^2 - 6a + 2)x - 3a^3 + 6a^2 - 2a$

$y = (3a^2 - 6a + 2)x - 2a^3 + 3a^2$

$y = (3a^2 - 6a + 2)x - a^2(2a - 3)$ .... (1)



(b)

since the tangent passes through $(0, p)$, substitute $(0, p)$ into (1)

$p = -2a^3 + 3a^2$

Since three tangents passes through that point, then the equation $-2a^3 + 3a^2 - p = 0$ must have 3 roots for a.

Let $g(a) = -2a^3 + 3a^2 - p$ then $g'(a) = -6a^2 + 6a = -6a(a - 1)$

This means the equation $-2a^3 + 3a^2 - p = 0$ has a minimum point at $a = 0$ and a maximum point at $a = 1$

$g(0) = -p$ has to be less than 0 and ...... the minimum must be less than 0

$g(1) = -2 + 3 - p$ has to be more than 0 ..... the maximum must be more than 0

i.e. $-p < 0 \Rightarrow p > 0$ and $1 - p > 0 \Rightarrow p < 1$

As such the range is $0 < p < 1$