Rational Functions:

Asymptotes

Consider a rational function, $f(x)$, in lowest terms of the form:

$$f(x) = \frac{P(x)}{Q(x)}$$

where the degree of $Q(x)$ is greater than or equal to 1. Then the asymptotes of $f(x)$ can be defined as follows:

Vertical Asymptote

A vertical asymptote occurs at each root of $Q(x) = 0$. The equation of the asymptotes are in the form $x = a$, where a is a root of $Q(x) = 0$.

Horizontal Asymptote

If the degree of $P(x)$ is less than the degree of $Q(x)$, then there is a horizontal asymptote at $y = 0$ (the x-axis).

If the degree of $P(x)$ is equal to the degree of $Q(x)$, then there is a horizontal asymptote at $y = \frac{p}{q}$, where p and q are the leading coefficients of $P(x)$ and $Q(x)$, respectively.

Oblique Asymptotes/ Slant Asymptotes and Others

If the degree of $P(x)$ is one greater than the degree of $Q(x)$, then there is an oblique or slant asymptote. If the degree of $P(x)$ is higher than the degree of $Q(x)$ by 2 or more, then the asymptote will be a curve. (This is rarely covered, and you are probably not responsible for it in your course.)

To find out the equation of the asymptote, perform the polynomial division $P(x) \div Q(x)$ and write the result as "quotient + $\frac{\text{remainder}}{\text{divisor}}$". The quotient is the equation of the asymptote.

The graph of a rational function can cross a non-vertical asymptote, but it cannot cross a vertical asymptote. (Non-vertical asymptotes only describe what a function does as it goes to $\pm\infty$, so for values somewhat close to 0, a function's graph can cross the asymptote.)

Note that a rational function has as many vertical asymptotes as its denominator has roots. (Double roots don't count twice, however.)

Notice, too, that a function can only have a horizontal asymptote or an oblique/slant asymptote, but not both (since the degree of $P(x) \le$ degree of $Q(x)$, or the degree of $P(x) >$ degree of $Q(x)$, but never both). If you find one of these types of asymptote, there is no need to waste time looking for the one.

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