Exercise 3.5. Supersymmetry. (1)

a) $\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu \chi + F^* F$

$$\begin{cases} \delta\phi = -i\,\varepsilon^T \sigma^2 \chi \\ \delta\chi = \varepsilon F + \sigma^\mu (\partial_\mu \phi)\sigma^2 \varepsilon^* \\ \delta F = -i\,\varepsilon^\dagger \bar{\sigma}^\mu (\partial_\mu \chi) \end{cases}$$

$$\begin{aligned}\mathcal{L} \to\ & \partial_\mu (\phi^* + i\,\varepsilon^\dagger \sigma^2 \chi^*)\, \partial^\mu (\phi - i\,\varepsilon^T \sigma^2 \chi) \\ & + (\chi + \varepsilon F + \sigma^\nu (\partial_\nu \phi)\sigma^2 \varepsilon^*)^\dagger i \bar{\sigma}^\mu \partial_\mu (\chi + \varepsilon F + \sigma^\nu \partial_\nu \phi\, \sigma^2 \varepsilon^*) \\ & + (F - i\,\varepsilon^\dagger \bar{\sigma}^\nu \partial_\nu \chi)^* (F - i\,\varepsilon^\dagger \bar{\sigma}^\mu \partial_\mu \chi)\end{aligned}$$

Working out the first order part of this expression in ε:

$$\begin{aligned}\delta\mathcal{L} =\ & i\,\varepsilon^\dagger \sigma^2 (\partial_\mu \chi^*)(\partial^\mu \phi) - (\partial_\mu \phi^*)\, i\,\varepsilon^T \sigma^2 (\partial^\mu \chi) \\ & + \varepsilon^\dagger F^* i \bar{\sigma}^\mu \partial_\mu \chi + \varepsilon^T \sigma^{2\dagger} (\partial_\nu \phi^*) \sigma^{\nu\dagger} i \bar{\sigma}^\mu \partial_\mu \chi \\ & + \chi^\dagger i \bar{\sigma}^\mu \partial_\mu (\varepsilon F) + \chi^\dagger i \bar{\sigma}^\mu \sigma^\nu (\partial_\nu \partial_\mu \phi)\sigma^2 \varepsilon^* \\ & - i F^* \varepsilon^\dagger \bar{\sigma}^\mu \partial_\mu F + i\,\varepsilon^T \bar{\sigma}^{*\nu} (\partial_\nu F^*)\, F\end{aligned}$$

$$\begin{aligned}\delta\mathcal{L} =\ & i\,\varepsilon^\dagger \sigma^2 (\partial_\mu \chi^*)(\partial^\mu \phi) - i\,\varepsilon^T \sigma^2 \underline{(\partial^\mu \chi)(\partial_\mu \phi^*)} \\ & + i\,\varepsilon^\dagger \cancel{F^* \bar{\sigma}^\mu} (\partial_\mu \chi) + i\,\varepsilon^T \sigma^2 \sigma^\nu \bar{\sigma}^\mu \underline{(\partial_\mu \chi)(\partial_\nu \phi^*)} \\ & + i \chi^\dagger \bar{\sigma}^\mu \varepsilon (\partial_\mu F) + i \chi^\dagger \bar{\sigma}^\mu \sigma^\nu (\partial_\nu \partial_\mu \phi)\sigma^2 \varepsilon^* \\ & - i \cancel{F^* \varepsilon^\dagger} \bar{\sigma}^\mu \partial_\mu \chi - i\,\varepsilon^T \bar{\sigma}^{\nu *} (\partial_\nu \chi^*)\, F\end{aligned}$$

$$\begin{aligned}\delta\mathcal{L} =\ & i\,\varepsilon^\dagger \sigma^2 (\partial_\mu \chi^*)(\partial^\mu \phi) + i\,\varepsilon^T \sigma^2 (\sigma^\nu \bar{\sigma}^\mu - g^{\mu\nu})(\partial_\mu \chi)(\partial_\nu \phi^*) \\ & + i \chi^\dagger \bar{\sigma}^\mu \varepsilon (\partial_\mu F) - i\,\varepsilon^T \bar{\sigma}^{\nu *} (\partial_\nu \chi^*)\, F \\ & + i \chi^\dagger \bar{\sigma}^\mu \sigma^\nu (\partial_\nu \partial_\mu \phi)\sigma^2 \varepsilon^* \\ & \quad \hookrightarrow i\,\varepsilon^\dagger \sigma^2 \chi^* (\partial^\mu \partial_\mu \phi)\end{aligned}$$

$$\begin{aligned}\delta\mathcal{L} =\ & i\,\varepsilon^\dagger \sigma^2 (\partial_\mu \chi^*)(\partial^\mu \phi) + i\,\varepsilon^\dagger \sigma^2 \chi^* (\partial^\mu \partial_\mu \phi) \\ & - i\,\varepsilon^T \bar{\sigma}^{*\mu} \chi^* (\partial_\mu F) - i\,\varepsilon^T \bar{\sigma}^{*\mu} (\partial_\mu \chi^*)\, F \\ & + i\,\varepsilon^\dagger \sigma^2 \chi^* (\partial^\mu \partial_\mu \phi)\end{aligned}$$