

$$\begin{aligned}
\text{ad}_x(y) &= d(\text{Ad}_e)_x(y) \\
&= \lim_{\varepsilon \rightarrow 0} \frac{(I + \varepsilon x)y(I + \varepsilon x)^{-1} - y}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{(I + \varepsilon x)y(I - \varepsilon x + (\varepsilon x)^2 + O(\varepsilon^3)) - y}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{((I + \varepsilon x)yI - (I + \varepsilon x)y\varepsilon x + (I + \varepsilon x)y(\varepsilon x)^2 + O(\varepsilon^3)) - y}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{(IyI + \varepsilon xyI - Iy\varepsilon x - \varepsilon xy\varepsilon x + Iy(\varepsilon x)^2 + \varepsilon xy(\varepsilon x)^2 + O(\varepsilon^3)) - y}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} \frac{y + xy\varepsilon - yx\varepsilon - xyx\varepsilon^2 + yx^2\varepsilon^2 + xyx^2\varepsilon^2 + O(\varepsilon^3) - y}{\varepsilon} \\
&= \lim_{\varepsilon \rightarrow 0} xy - yx - xyx\varepsilon + yx^2\varepsilon + xyx^2\varepsilon + O(\varepsilon^2) \\
&= [x, y]
\end{aligned}$$