

6. Let

$$A = \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & 1 \end{pmatrix}.$$

Find the eigenvalues of  $A$  and prove that  $A$  is diagonalizable, i.e., there is an invertible matrix  $P$  such that  $P^{-1}AP$  is diagonalizable. Furthermore,  $P$  can be chosen to be an orthogonal matrix, i.e., a matrix whose columns are unit vectors and whose distinct columns are orthogonal to each other.

$$\chi_A(t) = \det(A - tI) = \begin{vmatrix} 1-t & -\frac{3}{2} \\ -\frac{3}{2} & 1-t \end{vmatrix} = (1-t)^2 - \frac{9}{4}$$

Roots are:  $-\frac{1}{2}$  and  $\frac{5}{2}$ .

Notice also:  $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ ,  $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{5}{2} \\ -\frac{5}{2} \end{bmatrix} = \frac{5}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ .

$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$  is an orthogonal matrix,  $P^{-1}AP = \begin{bmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{5}{2} \end{bmatrix}$

Recall: Thm If  $A$  is a symmetric  $n \times n$  matrix, then  $A$  is diagonalizable by an orthogonal matrix  $P$ , i.e.  $P^{-1}AP$  is diagonal. Equivalently,  $\mathbb{R}^n$  has an ~~orthogonal~~ orthogonal basis of eigenvectors for  $A$ .

7. Let  $A$  be a  $3 \times 3$  symmetric matrix with real entries, and have eigenvalues  $\lambda_1 = 0$ ,  $\lambda_2 = 6$ , and  $\lambda_3 = 3$ . Let  $\mathbf{u}_1$ ,  $\mathbf{u}_2$ , and  $\mathbf{u}_3$  be corresponding eigenvectors (normalized to have length one).

(a) Since  $A$  is symmetric and  $\lambda_1$ ,  $\lambda_2$ , and  $\lambda_3$  are all different, it follows that

$$\mathbf{u}_1 \cdot \mathbf{u}_2 = \mathbf{u}_1 \cdot \mathbf{u}_3 = \mathbf{u}_2 \cdot \mathbf{u}_3 = 0$$

(b) The  $3 \times 3$  matrix  $P = [\mathbf{u}_1 \mid \mathbf{u}_2 \mid \mathbf{u}_3]$  satisfies

$$P^T P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (\text{fill in the entries of this } 3 \times 3 \text{ matrix}).$$

(c) Let  $P$  be the matrix of normalized eigenvectors from (b). Then  $A = PDP^T$ , where

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad (\text{fill in the entries of this } 3 \times 3 \text{ matrix}).$$

(d) The characteristic polynomial of  $A$  is  $(-t) \cdot (6-t) \cdot (3-t)$ .